Chapter 1

COORDINATED FLEXIBLE DISTRIBUTED MODEL PREDICTIVE CONTROL

Yi ZHENG[*]***, Shaoyuan LI***
School of Electronic Information and Electrical Engineering,
Shanghai Jiao Tong University, Shanghai, 200240, China

PACS 05.45-a, 52.35.Mw, 96.50.Fm. **Keywords:** Predictive Control, distributed systems, distributed model predictive control, networked control. **A**MS Subject Classification: 53D, 37C, 65P.

Abstract

Consider a complex large-scale control system which is composed of many spatially distributed subsystems. Each subsystem interacts with some other subsystems by their states and/or inputs, e.g. large-scale chemical process, smart grid, distributed generation systems. The control objective is to achieve a specific global performance of the entire system or a common goal of all subsystems.

To control this class of system, the Distributed Model Predictive Control (DMPC), which controls each subsystem by a separate local Model Predictive Control (MPC), has become more and more popular since it not only inherits MPC's ability to explicitly accommodate constraints but also possesses the advantages of the distributed framework of good flexibility and good error tolerance. On the other hand, with the he development of communication network technologies in process industries, which allow a distributed controller to access and send information throughout the system, also helps to promote distributed control solutions. However, as point in many articles, the performance of a DMPC is, in most cases, not as good as that of a centralized MPC.

The flexibility (or error tolerance) and global performance is two important characteristics of a DMPC. To improve the optimization performance, the existing methods usually increase the coordination degree (the range of cost that each subsystem-based MPC minimized). With the increasing of the coordination degree, the performance of entire system becomes better and better. However, with the increasing of the coordination degree, the network connectivity become more and more complicity, and consequently the error tolerance and high flexibility become weaker and weaker. It is

[*]E-mail address: yizheng@sjtu.edu.cn

not expected. Can we find a method which could improve the global performance or coordination degree without any increasing of network connectivity?

In this Chapter, a novel coordination strategy, where each subsystem-based model predictive control (MPC) added a quadratic function of the affection of the current subsystems input to its down-stream neighbors into its optimization index, is proposed for improving the optimization performance of entire system. This method is able to increase the coordination degree without any increasing of network connections comparing to the methods which do not use this coordination strategy. The consistency constraints, which limit the error between the state predicted at the previous time instant, referred to as the presumed state, and the state predicted at the current time instant within a prescribed bound, are designed and included in the optimization problem of each subsystem-based MPC. These constraints guarantee the recursive feasibility of each subsystem-based MPC. In the meantime, a stabilization constraint and the dual mode predictive control strategy are adopted to result in a stabilizing DMPC.

1. Introduction

Consider a complex large-scale control system which is composed of many physically or geographically divided subsystems. Each subsystem interacts with some other subsystems by their states and/or inputs, e.g. large-scale chemical process [7], smart grid [8], distributed generation systems. The control objective is to achieve a specific global performance of the entire system or a common goal of all subsystems.

In controlling such a large-scale system, the distributed (or decentralized) framework, where each subsystem is controlled by an independent controller, is usually adopted despite the resulting global performance is in general not as good as a centralized solution [28, 27, 14] for several reasons. The classical centralized control solution is often impractical because of its heavy computational demand and its lack of tolerance to faults in controller. The integrity of the control system cannot be maintained when a control component fails and the whole system is out of control when the centralized controller fails. The distributed framework, in contrast, has the advantages of fault-tolerance, less computation and being flexible to system structure. In the mean time, the development of communication network technologies in process industries, which allows a distributed controller to access and send information throughout the system, also helps to promote distributed control solutions [32].

Among the distributed solutions, the Distributed Model Predictive Control (DMPC), which controls each subsystem by a separate local Model Predictive Control (MPC), has become more and more popular [25, 7] since it not only inherits MPC's ability to explicitly accommodate constraints [24, 19, 17, 30, 35] but also possesses the advantages of the distributed framework mentioned above. However as pointed in [4, 28], the performance of a DMPC is, in most cases, not as good as that of a centralized MPC.

Many algorithms have appeared in the literature for different type of systems and for different problems in the design of DMPC, e.g. design of DMPC for nonlinear systems [2, 12], design DMPC for networked systems with time delay [18, 36], development of decentralized optimization algorithm for DMPC [6], and design cooperative strategy for improving performance of DMPC [33, 9], as well as the design of control structure of DMPC [1]. Among them, several coordination strategies have appeared to improve the global performance of the DMPC, and can be classified according to the information exchange protocol needed (i.e non-iterative or iterative algorithms), and to the type of cost function which is optimized [25]. The DMPCs, which accommodate same kind of cost function for each subsystem-based MPC, can be solved either by iterative algorithm or non-iterative algorithm. Thus the coordination strategies are introduced by the classification of the type of cost function which is optimized in each subsystem-based MPC. The simplest and most adopted strategy is that each local controller minimizes its own subsystem's cost and uses the state prediction of the previous time instant to approximate the state sequence at the current time instant in computing the optimal solution [4, 15, 10]. In this kind of method, each subsystem based controller should communicate with its neighbors. Another commonly used coordination strategy is that each subsystem-based MPC optimizes the cost of overall system to improve the global performance [28, 26, 33, 9, 31]. In computing the optimal solution, it also uses the state prediction of the previous time instant to approximate the state sequence at the current time instant. This strategy could achieve a good global performance in some cases, but it reduces the flexibility and increases the communication load since each subsystem-based MPC should communicate with all other subsystems. In an effort to achieve a trade off between the global performance of the entire system and the computational burden, recently, an intuitively appealing strategy is proposed in [34, 32, 36, 16], where each subsystem-based MPC only considers the cost of its own subsystem and those of the subsystems it directly impacts on. Such a design can be referred to as *Impacted-region Cost Optimization based DMPC* (ICO-DMPC). In particular, Ref. [34, **?**] applies this design idea to a metallurgy system and [32] explains why this coordination strategy could improve the global performance. Numerical and practical experiments show that this coordination strategy could obtain a performance close to that of a classical centralized M-PC. In this kind of method, each subsystem based controller should communicate with its neighbors and its neighbours' neighbours.

It can be seen from above that, in existing methods, with the increasing of the coordination degree, the performance of entire system becomes better and better, the network connectivity becomes more and more complicity, and consequently the error tolerance and high flexibility become weaker and weaker. To find a method which could improve the global performance or coordination degree without any increasing of network connectivity is still remain to be solved.

In this Chapter, an Coordinated Flexible Distributed Model Predictive Control (CF-DMPC) is proposed and presented for distributed systems, where each subsystem-based MPC adds a quadratic function of the impact of current subsystem's input to its down stream neighbours into its optimization index to increase the coordination degree. It do not need any additional network connectivity comparing to the approach without this coordination strategy. The consistency constraints, which limit the error between the state predicted at the previous time instant, referred to as the presumed state, and the state predicted at the current time instant within a prescribed bound, are designed and included in the optimization problem of each subsystem-based MPC. These constraints can bound the error between the presumed state and the predictive state of upstream neighbors and the error between the predictive state of downstream neighbor calculated by current subsystem-based MPC and that of the downstream neighbors themselves. And these constraints guarantee the feasibility of each subsystem-based MPC. They also guarantee that the remaining part of the solution at the previous time instant is a feasible solution. In the mean time, a stabilization

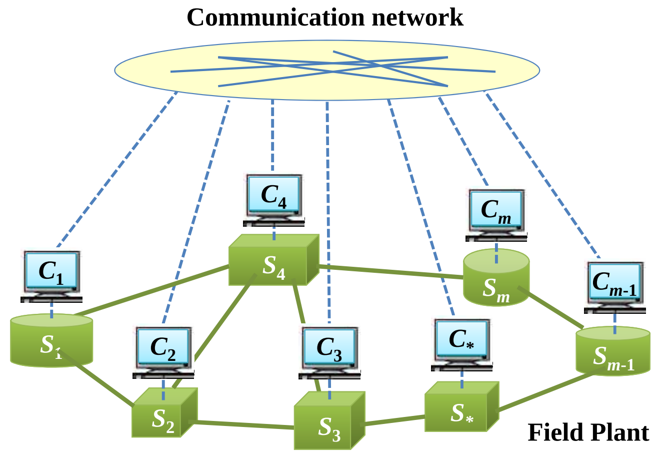

Figure 1. An illustration of the structure of a distributed system and its distributed control framework.

constraint and the dual mode predictive control [13, 23] strategy are adopted to result in a stabilizing ICO-DMPC.

The remainder of this Chapter is organized as follows. Section 2. describes the problem to be solved in this paper. Section 3. presents the design of the stabilizing CF-DMPC. The feasibility of the proposed CF-DMPC and the stability of the resulting closed-loop system are analyzed in Section 4.. Section 5. presents the simulation results to demonstrate the effectiveness of the proposed DMPC algorithm. Finally, a brief conclusion to the paper is drawn in Section 6..

2. Problem Description

2.1. Distributed Systems

A distributed system, as illustrated in Fig. 1, is composed of many interacting subsystem-s, each of which is controlled by a subsystem-based controller, which in turn is able to exchange information with other subsystem-based controllers.

Suppose that the distributed system $\mathcal{S}$ is composed of m discrete-time linear subsystems $\mathcal{S}_i$, $i \in \mathcal{P} = \{1, 2, \cdots, m\}$ and m controllers $\mathcal{C}_i$, $i \in \mathcal{P} = \{1, 2, \cdots, m\}$. Let the subsystems interact with each other through their states. If subsystem $\mathcal{S}_i$ is affected by $\mathcal{S}_j$, for any $i \in \mathcal{P}$ and $j \in \mathcal{P}$, subsystem $\mathcal{S}_i$ is said to be a downstream system of subsystem $\mathcal{S}_j$, and subsystem $\mathcal{S}_j$ is an upstream system of $\mathcal{S}_i$. Let $\mathcal{P}_{+i}$ denote the set of the subscripts of the upstream systems of $\mathcal{S}_i$, $\mathcal{P}_{-i}$ is the set of the subscripts of the downstream systems of $\mathcal{S}_i$. Then, subsystem $\mathcal{S}_i$ can be expressed as

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i,k+1} & = & \mathbf{A}_{ii}\mathbf{x}_{i,k} + \mathbf{B}_{ii}\mathbf{u}_{i,k} + \sum\limits_{j \in \mathcal{P}_{+i}} \mathbf{A}_{ij}\mathbf{x}_{j,k}, \\ \mathbf{y}_{i,k} & = & \mathbf{C}_{ii}\mathbf{x}_{i,k}, \end{array} \right. \tag{1}$$

where $\mathbf{x}_i \in \mathbf{R}^{n_{xi}}$, $\mathbf{u}_i \in \mathcal{U}_i \subset \mathbf{R}^{n_{ui}}$ and $\mathbf{y}_i \in \mathbf{R}^{n_{yi}}$ are respectively the local state, input and output vectors, and $\mathcal{U}_i$ is the feasible set of the input $\mathbf{u}_i$, which is used to bound the input according to the physical constraints on the actuators, the control requirements or the characteristics of the plant. A non-zero matrix $\mathbf{A}_{ij}$, that is, $j \in \mathcal{V}_i^{\sim}$, indicates that $\mathcal{S}_i$ is affected by $\mathcal{S}_j$. In the concatenated vector form, the system dynamics can be written as

$$\begin{cases} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k, \end{cases} \tag{2}$$

where $\mathbf{x} = [\mathbf{x}_1^{\mathrm{T}} \ \mathbf{x}_2^{\mathrm{T}} \ \cdots \ \mathbf{x}_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{n_x}$, $\mathbf{u} = [\mathbf{u}_1^{\mathrm{T}} \ \mathbf{u}_2^{\mathrm{T}} \ \cdots \ \mathbf{u}_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{n_u}$ and $\mathbf{y} = [\mathbf{y}_1^{\mathrm{T}} \ \mathbf{y}_2^{\mathrm{T}} \ \ldots \ \mathbf{u}_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{n_y}$ are respectively the concatenated state, control input and output vectors of the overall system $\mathcal{S}$, and A, B and C are constant matrices of appropriate dimensions. Also, $\mathbf{u} \in \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_m$.

The control objective is to stabilize the overall system $\mathcal{S}$ in an DMPC framework. Meanwhile, the achieved performance index of the overall system should be as close as possible to the performance index achievable under a centralized MPC and the network connectivity should not increase.

2.2. Existing Methods and Motivations

The works on DMPCcan be roughly divided in optimization-based frameworks which focus on the solution of the optimization problems [21]. and control- based frameworks which are concerned mostly with stability and control performance [5].

The control-based frameworks can be classified according to the information exchange protocol needed (i.e non-iterative or iterative algorithms), and to the type of cost function which is optimized. The non-iterative algorithms which only communicates once a control period have faster computational speed comparing to that of iterative algorithm and the iterative algorithms could obtains more excellent optimization performance than non-iterative algorithms. The DMPC which accommodate same kind of cost function for subsystem-based MPC can be solved either by iterative algorithm or non-iterative algorithm. Different methods have different characteristics and are suitable for different control purposes. We briefly review these methods as motivations for the problem we have just formulated and its solution to be presented later in the paper.

1) Distributed algorithms where each subsystem-based controller minimizes the cost function of its own subsystem were proposed in [4, 15],

$$J_{i,k} = \|\mathbf{x}_{i,k+N}\|_{\mathbf{P}_i}^2 + \sum_{l=0}^{N-1} \left(\|\mathbf{x}_{i,k+l}\|_{\mathbf{Q}_i}^2 + \|\mathbf{u}_{i,k+l}\|_{\mathbf{R}_i}^2 \right) \tag{3}$$

When computing the optimal solution, each local controller exchanges state estimation with the neighboring subsystems to improve the performance of the local subsystem. This method is simple and very convenient for implementation. An extension of this stabilizing DMPC with input constraint for nonlinear continuous systems is given in [10]. In addition [11] gives a design for linear systems, which could handle both input and states constraints by using a fixed reference state trajectory with moving widow to replace the state estimation in each update.

2) To improve the global performance, distributed algorithms, where each local controller minimizes a global cost function

$$\widetilde{J}_{i,k} = \sum_{j \in \mathcal{P}} J_{j,k}, \tag{4}$$

were proposed in [28, 33, 26, 9, 31]. In this method, each subsystem-based MPC exchanges information (the estimation of the future state sequences of the subsystems) with all other subsystems. And some iterative stabilizing designs are proposed which take the advantages of the model of whole system is used in each subsystem-based MPC. This strategy may result in a better performance but consumes much more communication resources, in comparison with the method in described in 1).

3) To balance the performance, communication cost and the complexity of the DMPC algorithm, the strategy that each subsystem-based controller only minimizes its own cost function and those of the subsystems that its own subsystem directly impacts on was recently proposed in [34, 32, 36, 16], that is

$$\bar{J}_{i,k} = \sum_{j \in \mathcal{P}_i} J_{j,k}, \tag{5}$$

where $\mathcal{P}_i = \{j : j \in \mathcal{P}_{-i} \text{ or } j = i\}$ is the set of subscripts of the downstream subsystems of subsystem $\mathcal{S}_i$, that is the region impacted on by subsystem $\mathcal{S}_i$. The resulting control algorithm is termed as an Impacted-region Cost Optimization based DMPC, or ICO-DMPC. It could achieve a better performance than the first method, and its communication burden is much less than the second method [1, 3].

Clearly, this coordination strategy as proposed in [34, 32] and described in 3) is a preferable method to trade-off the communication burden and the global performance. However, the DMPC under this coordination strategy the network conductivities is still increase, although the global performance is dramatically increased with less increasing of network connectivity. A method which could increase the coordination degree of DMPC without increasing of network connectivity remains to be developed. The objective of this Chapter is to develop and introduce such a DMPC design.

3. Coordinated Flexible Distributed Model Predictive Control Formulation

In this Section, m separate optimal control problems, one for each subsystem, and the Coordinated Flexible Distributed Model Predictive Control (CF-DMPC) which coordinates the subsystem-based MPCs by adding a quadratic function of the impact of the current subsystem's input to its down stream neighbours into its optimization index is defined. In every distributed optimal control problem, the same constant prediction horizon N, $N > 1$, is used. And every distributed MPC law is updated globally synchronously. At each update, every local MPC optimizes only for its own open-loop control sequence, given the current states and the estimated inputs of the whole system.

To proceed, we need the following assumption,

Table 1. Notation

Notation	Explanation
$\mathcal{P}$	the set of the subscripts of all subsystems
$\mathcal{P}_i$	the set of the subscripts of all subsystems, excluding $\mathcal{S}$ itself
$+i$	the subscript denote all upstream neighbors of $\mathcal{S}_i$
$-i$	the subscript denote all downstream neighbors of $\mathcal{S}_i$
$\mathbf{u}_{i,k+l-1\|k}$	the optimal control sequence of $\mathcal{S}_i$ calculated by $\mathcal{C}_i$ at time k
$\hat{\mathbf{x}}_{i,k+l\|k}$	the presumed state sequence of $\mathcal{S}_i$ define by $\mathcal{C}_i$ at time k
$\hat{\mathbf{u}}_{i,k+l\|k}$	the presumed input sequence of $\mathcal{S}_i$ define by $\mathcal{C}_i$ at time k
$\mathbf{u}^{\mathrm{f}}_{i,k+l-1\|k}$	the feasible control at time k+l-1 of $\mathcal{S}_i$ defined by $\mathcal{C}_i$ at time k
$\mathbf{x}^{\mathrm{f}}_{i,k+l\|k}$	the predictive feasible state sequence of $\mathcal{S}_i$ defined by $\mathcal{C}_i$ at time k
$\mathbf{u}^{\mathrm{p}}_{i,k+l-1\|k}$	the predicted control at time k+l-1 of $\mathcal{S}_i$ defined by $\mathcal{C}_i$ at time k
$\mathbf{x}^{\mathrm{p}}_{i,k+l\|k}$	the predicted state sequence of $\mathcal{S}_i$ defined by $\mathcal{C}_i$ at time k
$\|\|\cdot\|\|_{\mathbf{P}}$	refer to the P norm, P is any positive matrix, and $\|\|\mathbf{z}\|\|_{\mathbf{P}} = \sqrt{\mathbf{x_k^{\mathrm{T}}}\mathbf{P}\mathbf{x_k}}$.

Assumption 1 *For every subsystem $\mathcal{S}_i$, $i \in \mathcal{P}$, there exists a decoupled static feedback $\mathbf{u}_i = \mathbf{K}_i\mathbf{x}_i$ such that $\mathbf{A}_{\mathrm{d}i} = \mathbf{A}_{ii} + \mathbf{B}_{ii}\mathbf{K}_i$ is Shur stable, and the closed-loop system $\mathbf{x}_{k+1} = \mathbf{A}_{\mathrm{c}}\mathbf{x}_k$ is asymptotically stable, where $\mathbf{A}_{\mathrm{c}} = \mathbf{A} + \mathbf{B}\mathbf{K}$ and $\mathbf{K} = \mathrm{diag}\{\mathbf{K}_1, \mathbf{K}_2, \cdots, \mathbf{K}_m\}$.*

This assumption is usually used in the design of stabilizing DMPC, see [11, 10]. It presumes that each subsystem is able to be stabilized by a decentralized control $\mathbf{K}_i\mathbf{x}_i$, $i \in \mathcal{V}$. and the decentralized control gain $\mathbf{K}$ based on LMI have been proposed in [29] for continuous time systems. These techniques can be easily adapted to the discrete-time case here considered.

We also define the necessary notation in Table 1.

Consider that the control law of current subsystem $\mathcal{S}_i$ effects the performance of its downstream neighboring subsystems $\mathcal{S}_j$, $j \in \mathcal{P}_{-i}$, in the CF-DMPC, the performance of $\mathcal{S}_j$ is added into the performance index of the MPC which control $\mathcal{S}_i$ based on a approximation of the updated state sequence of $\mathcal{S}_j$. The approximated state sequence equals the assumed state sequence of $\mathcal{S}_j$ pluses the impact caused by the change of control law of $\mathcal{S}_i$ to the state sequence of $\mathcal{S}_j$. In this way, the coordination degree is expanded without any increasing of the required network connectivity in solving each subsystem-based MPC.

Define that $\mathbf{f}_{i,k+l|k}$ be the matching from $\mathbf{u}_{i,k:k+l-1|k}$ to $\mathbf{x}_{i,k+l|k}$, and it can be deduced from equation (1) as

$$\begin{aligned}\mathbf{f}_{i,k+l|k} &= \mathbf{x}_{i,k+l|k}\\ &= \mathbf{A}_{ii}^{l}\mathbf{x}_{i,k} + \sum_{h=1}^{l}\mathbf{A}_{ii}^{l-h}\mathbf{B}_{ii}\mathbf{u}_{i,k+h-1|k}\\ &\quad + \sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}_{ii}^{l-h}\mathbf{A}_{ij}\mathbf{x}_{j,k+h-1|k} \end{aligned} \tag{6}$$

Then, it have

$$\frac{\partial \mathbf{f}_{i,k+l|k}}{\partial \mathbf{x}_{j,k+h-1|k}} = \mathbf{A}_{ii}^{l-h}\mathbf{A}_{ij} \tag{7}$$

$$\frac{\partial \mathbf{x}_{i,k+l|k}}{\partial \mathbf{u}_{i,k+h-1|k}} = \mathbf{A}_{ii}^{l-h}\mathbf{B}_{ii} \tag{8}$$

The $\mathbf{f}_{i,k+l|k}$ derivation of the $\mathbf{u}_{j,k+h-1|k}$ becomes

$$\begin{aligned}\frac{\partial \mathbf{f}_{i,k+l|k}}{\partial \mathbf{u}_{j,k+h-1|k}} &= \sum_{p=h+1}^{l}\frac{\partial \mathbf{f}_{i,k+l|k}}{\partial \mathbf{x}_{j,k+p-1|k}}\frac{\partial \mathbf{x}_{j,k+p-1|k}}{\partial \mathbf{u}_{j,k+h-1|k}}\\ &= \sum_{p=h+1}^{l}\mathbf{A}_{ii}^{l-p}\mathbf{A}_{ij}\mathbf{A}_{jj}^{p-h}\mathbf{B}_{jj}. \end{aligned} \tag{9}$$

Since the state and input sequences of downstream and upstream neighbors of $\mathcal{S}_i$ is unknown to the controller of $\mathcal{S}_i$, assume that the state and input sequences $\hat{\mathbf{x}}_{i,k+l|k}$ and $\hat{u}_{i,k+l|k}$ be the presumed states and presumed input which are calculated in the previous calculation, respectively. Add the estimation of the performance of the $\mathcal{S}_j$, $j \in \mathcal{P}_{-i}$ to the cost function of the MPC for $\mathcal{S}_i$, then the optimization index of $\mathcal{S}_i$ becomes

$$\begin{aligned}\bar{J}_i(k) &= \sum_{l=1}^{N}\left(\left\|\mathbf{x}^{\mathrm{p}}_{i,k+l|k}\right\|^2_{\mathbf{Q}_i} + \left\|\mathbf{u}_{i,k+l-1|k}\right\|^2_{\mathbf{R}_i}\right)\\ &\quad + \sum_{j\in\mathcal{P}_{-i}}\sum_{l=1}^{N}\left\|\left(\hat{\mathbf{x}}_{j,k+l|k} + \omega_i\mathbf{S}_{ji,k+l|k}\right)\right\|^2_{\mathbf{Q}_j} + \sum_{j\in\mathcal{P}_{-i}}\sum_{l=1}^{N}\left\|\hat{\mathbf{u}}_{i,k+l-1|k}\right\|^2_{\mathbf{R}_j} \end{aligned} \tag{10}$$

where ω_i is the weighting coefficients for improving the convergence when using iterative algorithm, and

$$\begin{aligned}&\mathbf{S}_{ji,k+l|k} = \sum_{h=1}^{l}\sum_{p=h+1}^{l}\mathbf{A}_{jj}^{l-p}\mathbf{A}_{ji}\mathbf{A}_{ii}^{p-h}\mathbf{B}_{ii}(\mathbf{u}_{i,k+h-1|k} - \hat{\mathbf{u}}_{i,k+h-1|k})\\ &h = 1, 2, \ldots, l. \end{aligned} \tag{11}$$

where $\mathbf{Q}_i = \mathbf{Q}_i^{\mathrm{T}} > 0$, $\mathbf{R}_i = \mathbf{R}_i^{\mathrm{T}} > 0$ and $\mathbf{P}_j = \mathbf{P}_j^{\mathrm{T}} > 0$. The matrix $\mathbf{P}_i$ is chosen to satisfy the Lyapunov equation

$$\mathbf{A}_{\mathrm{d}i}^{\mathrm{T}}\mathbf{P}_i\mathbf{A}_{\mathrm{d}i} - \mathbf{P}_i = -\widehat{\mathbf{Q}}_i,$$

where $\widehat{\mathbf{Q}}_i = \mathbf{Q}_i + \mathbf{K}_i^{\mathrm{T}}\mathbf{R}_i\mathbf{K}_i$. Denote

$$\begin{aligned}
\mathbf{P} &= \mathrm{diag}\{\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_m\}, \\
\mathbf{Q} &= \mathrm{diag}\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_m\}, \\
\mathbf{R} &= \mathrm{diag}\{\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_m\}, \\
\mathbf{A}_{\mathrm{d}} &= \mathrm{diag}\{\mathbf{A}_{\mathrm{d}1}, \mathbf{A}_{\mathrm{d}2}, \cdots, \mathbf{A}_{\mathrm{d}m}\}.
\end{aligned}$$

Then, it follows that

$$\mathbf{A}_{\mathrm{d}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{P} = -\widehat{\mathbf{Q}},$$

where $\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{K}^{\mathrm{T}}\mathbf{R}\mathbf{K} > 0$.

Since every subsystem-based controller is updated synchronously, the state and control sequences of other subsystems are unknown to subsystem $\mathcal{S}_i$. Thus, at the time instant k, the presumed state sequence $\hat{\mathbf{x}}_{j,k:k+N|k}$ of $\mathcal{S}_j$ are used in the predictive model of the MPC in $\mathcal{S}_i$, which is given as

$$\begin{aligned}
\mathbf{x}^{\mathrm{p}}_{i,k+l|k} = {} & \mathbf{A}_{ii}^{l}\mathbf{x}^{\mathrm{p}}_{i,k} + \sum_{h=1}^{l}\mathbf{A}_{ii}^{l-h}\mathbf{B}_{ii}\mathbf{u}_{i,k+h-1|k} \\
& + \sum_{j\in\mathcal{P}_i^u}\sum_{h=1}^{l}\mathbf{A}_{ii}^{l-h}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k+h-1|k}
\end{aligned} \tag{12}$$

Given $\mathbf{x}^{\mathrm{p}}_{i,k|k} = \mathbf{x}_i(k|k)$, the presumed control sequence for subsystem $\mathcal{S}_i$ is given by

$$\hat{\mathbf{u}}_{i,k+l-1|k} = \begin{cases} \mathbf{u}^{\mathrm{p}}_{i,k+l-1|k-1}, l = 1, 2, \cdots, N-1, \\ \mathbf{K}_i\mathbf{x}^{\mathrm{p}}_{i,k+N-1|k-1}, l = N \end{cases} \tag{13}$$

Set each presumed state sequence $\hat{\mathbf{x}}_i$ to be the remainder of the sequence predicted at the previous time instant $k-1$, concatenated with the closed-loop response under the state feedback control $\hat{\mathbf{u}}_{i,k+l-1|k} = \mathbf{K}_j\mathbf{x}^{\mathrm{p}}_{i,k+N-1|k-1}$, that is that is

$$\begin{cases} \hat{\mathbf{x}}_{i,k+l-1|k} = \mathbf{x}^{\mathrm{p}}_{k+l-1|k-1}, l = 1, 2, \cdots, N, \\ \hat{\mathbf{x}}_{i,k+N+1-1|k} = \mathbf{A}_{\mathbf{d}i}\mathbf{x}^{\mathrm{p}}_{i,k+N-1|k-1} + \sum\limits_{j\in\mathcal{P}_{+i}}\mathbf{A}_{ij}\mathbf{x}^{\mathrm{p}}_{j,k+N-1|k-1} \end{cases} \tag{14}$$

In MPC systems, it is an important proposition to focus on the systems sequence feasibility and the stability. The same things happen in the distributed MPC systems. To enlarge the feasible region, a terminal state constraint is included in each subsystem-based MPC, which guarantees that the terminal controllers are stabilizing inside a terminal set. To define this terminal state set, we need to make an assumption and establish a technical lemma.

Assumption 2 *The block-diagonal matrix* $\mathbf{A}_{\mathrm{d}} = \mathrm{diag}\{\mathbf{A}_{\mathrm{d}1}, \mathbf{A}_{\mathrm{d}2}, \ldots, \mathbf{A}_{\mathrm{d}m}\}$ *and the off-diagonal matrix* $\mathbf{A}_{\mathrm{o}} = \mathbf{A}_{\mathrm{c}} - \mathbf{A}_{\mathrm{d}}$ *satisfy the following inequality*

$$\mathbf{A}_{\mathrm{o}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{o}} + \mathbf{A}_{\mathrm{o}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{d}} + \mathbf{A}_{\mathrm{d}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{o}} < \widehat{\mathbf{Q}}/2.$$

It along with Assumption 1, help with the design of the terminal set. This assumption quantifies the coupling between subsystems which is sufficiently weak to be controlled by the algorithm designed here.

Lemma 3..1 *Under Assumptions 1 and 2, for any positive scalar c the set*

$$\Omega(c) = \{\mathbf{x} \in \mathbf{R}^{n_x} : \|\mathbf{x}\|_{\mathbf{P}} \le c\}$$

is a positive invariant region of attraction for the closed-loop system $\mathbf{x}_{k+1} = \mathbf{A}_{\rm c}\mathbf{x}_k$. Additionally, there exists a small enough positive scalar ε such that $\mathbf{Kx}$ is in the feasible input set $\mathcal{U} \subset \mathbf{R}^{n_u}$ for all $\mathbf{x} \in \Omega(\varepsilon)$.

Proof: Consider the function $V(k) = \|\mathbf{x}_k\|_{\mathbf{P}}^2$. The time difference of $V(k)$ along the trajectories of the closed-loop system $\mathbf{x}_{k+1} = \mathbf{A}_{\rm c}\mathbf{x}_k$ can be evaluated as

$$\begin{aligned}
\Delta V_k &= \mathbf{x}_k^{\rm T}\mathbf{A}_{\rm c}^{\rm T}\mathbf{P}\mathbf{A}_{\rm c}\mathbf{x}_k - \mathbf{x}_k^{\rm T}\mathbf{P}\mathbf{x}_k \\
&= \mathbf{x}_k^{\rm T}(\mathbf{A}_{\rm d}^{\rm T}\mathbf{P}\mathbf{A}_{\rm d} - \mathbf{P} + \mathbf{A}_{\rm o}^{\rm T}\mathbf{P}\mathbf{A}_{\rm o} + \mathbf{A}_{\rm o}^{\rm T}\mathbf{P}\mathbf{A}_{\rm d} + \mathbf{A}_{\rm d}^{\rm T}\mathbf{P}\mathbf{A}_{o})\mathbf{x}_k \\
&\le -\mathbf{x}_k^{\rm T}\hat{\mathbf{Q}}\mathbf{x}_k + \frac{1}{2}\mathbf{x}_k^{\rm T}\hat{\mathbf{Q}}\mathbf{x}_k \\
&\le 0,
\end{aligned}$$

which holds for all $\mathbf{x}(k) \in \Omega(c) \setminus \{0\}$. This implies that all trajectories of the closed-loop system that starts inside $\Omega(c)$ will remain inside and converge to the origin.

The existence of an $\varepsilon > 0$ such that $K\mathbf{x} \in \mathcal{U}$ for all $\mathbf{x} \in \Omega(\varepsilon)$ follows from the fact that P is positive definite, which implies that the set $\Omega(\varepsilon)$ shrinks to the origin as ε decreases to zero. This completes the proof. □

In the optimization problem of each subsystem-based MPC, the terminal state constraint set for each $\mathcal{S}_i$ can then be set to be

$$\Omega_i(\varepsilon) = \left\{\mathbf{x}_i \in \mathbf{R}^{n_{xi}} : \|\mathbf{x}_i\|_{\mathbf{P}_i} \le \varepsilon/\sqrt{m}\right\}.$$

Clearly, if $\mathbf{x} \in \Omega_1(\varepsilon) \times \cdots \times \Omega_m(\varepsilon)$, then the decoupled controllers will stabilize the system at the origin, since

$$\|\mathbf{x}_i\|_{\mathbf{P}_i}^2 \le \frac{\varepsilon^2}{m},\ \forall i \in \mathcal{P},$$

implies that

$$\sum_{i \in \mathcal{P}} \|\mathbf{x}_i\|_{\mathbf{P}_i}^2 \le \varepsilon^2,$$

which in turn implies that $x \in \Omega(\varepsilon)$. Suppose that at some time k_0, $\mathbf{x}_{i,k_0} \in \Omega_i(\varepsilon)$ for every subsystem. Then, by Lemma 3..1, stabilization can be achieved if every $\mathcal{C}_i, i \in \mathcal{P}$, employs its decoupled static feedback controller $K_i\mathbf{x}_{i,k}$ after time instant k_0.

Thus, the objective of each subsystem-based MPC law is to drive the state of each subsystem $\mathcal{S}_i$ to the set $\Omega_i(\varepsilon)$. Once all subsystems have reached these sets, they switch to their decoupled controllers for stabilization. Such switching from an MPC law to a terminal controller once the state reaches a suitable neighborhood of the origin is referred to as the dual-mode MPC [23, 13]. For this reason, the DMPC algorithm we propose in this paper is a dual-mode DMPC algorithm.

In what follows, we formulate the optimization problem for each subsystem-based M-PC.

Problem 1 *Consider subsystem $\mathcal{S}_i$. Let $\varepsilon > 0$ be as specified in Lemma 3..1. Let the update time be $k \geq 1$. Given $\mathbf{x}_{i,k}$, $bf\hat{x}_{j,k+l|k}, l = 1, 2, \ldots, N$, $\forall j \in \mathcal{P}_{+i}$, find the control sequence $\mathbf{u}_{i,k+l|k} : \{0, 1, \cdots, N-1\} \to \mathcal{U}_i$ that minimizes*

$$\begin{aligned}\bar{J}_i(k) &= \sum_{l=1}^{N}\left(\left\|\mathbf{x}^{\mathrm{p}}_{i,k+l|k}\right\|^2_{\mathbf{Q}_i} + \left\|\mathbf{u}_{i,k+l-1|k}\right\|^2_{\mathbf{R}_i}\right) \\ &\quad + \sum_{j\in\mathcal{P}_{-i}}\sum_{l=1}^{N}\left\|\hat{\mathbf{x}}_{j,k+l|k} + \omega_i \mathbf{S}_{ji,k+l|k}\right\|^2_{\mathbf{Q}_j} + \sum_{j\in\mathcal{P}_{-i}}\sum_{l=1}^{N}\left\|\hat{\mathbf{u}}_{i,k+l-1|k}\right\|^2_{\mathbf{R}_j} \quad (15)\end{aligned}$$

Subject to the constraints:

$$(12), \quad (16)$$

$$\sum\nolimits_{s=1}^{l} \alpha_{l-s}||\mathbf{x}^{\mathrm{p}}_{i,k+s|k} - \hat{\mathbf{x}}_{i,k+s|k}||_2 \leq \frac{\xi\kappa\varepsilon}{2\sqrt{mm_1}}, l = 1, 2, \ldots, N-1 \quad (17)$$

$$\left\|\mathbf{x}^{\mathrm{p}}_{i,k+N|k} - \hat{\mathbf{x}}_{i,k+N|k}\right\|_{\mathbf{P}_i} \leq \frac{\kappa\varepsilon}{2\sqrt{m}}, \quad (18)$$

$$\left\|\mathbf{x}^{\mathrm{f}}_{i,k+l|k}\right\|_{\mathbf{P}_i} \leq \left\|\mathbf{x}^{\mathrm{p}}_{i,k+l|k}\right\|_{\mathbf{P}_i} + \frac{\varepsilon}{\mu N\sqrt{m}}, l = 1, 2, \cdots, N, \quad (19)$$

$$\mathbf{u}_{i,k+l|k} \in \mathcal{U}_i, l = 0, 1, \ldots, N-1, \quad (20)$$

$$\mathbf{x}^{\mathrm{p}}_{i,k+N|k} \in \Omega_i(\varepsilon/2) \quad (21)$$

In the constraints above,

$$m_1 = \max_{i\in\mathcal{P}}\{\text{number of elements in } \mathcal{P}_{+i}\}, \quad (22)$$

$$\alpha_l = \max_{i\in\mathcal{P}}\max_{j\in\mathcal{P}_i}\left\{\lambda^{\frac{1}{2}}_{\max}\left(\left(\mathbf{A}^l_{ii}\mathbf{A}_{ij}\right)^T\mathbf{P}_j\mathbf{A}^l_{ii}\mathbf{A}_{ij}\right)\right\}, l = 0, 1, \cdots, N-1, \quad (23)$$

The constant $0 < \kappa < 1$, $0 < \xi \leq 1$ are design parameters whose values will be chosen in the sequel. Both (17) and (18) are referred to as the consistency constraints, which require that each predictive sequence and control variables remain close to their presumed values. These constraints are keys to proving that $\mathbf{x}^{\mathrm{f}}_i$ is a feasible state sequence at each update.

Equation (19) will be utilized to prove that the CF-DMPC algorithm is stabilizing, where $\mu > 0$ is a design parameter whose value will be specified later. $\mathbf{x}^{\mathrm{f}}_{i,k+l|k}$ is a feasible state sequence, and equals to the solution of (12) under the initial state of $\mathbf{x}_{i,k}$, the feasible control sequence $\mathbf{u}^{\mathrm{f}}_{i,k+l-1|k}$ is defined by

$$\mathbf{u}^{\mathrm{f}}_{i,k+l-1|k} = \begin{cases} \mathbf{u}^{\mathrm{p}}_{i,k+l-1|k-1}, & l = 1, 2, \cdots, N-1, \\ \mathbf{K}_i\mathbf{x}^{\mathrm{f}}_{i,k+N-1|k}, & l = N. \end{cases} \quad (24)$$

It should be noticed that the terminal constraint in each optimal control problem is $\Omega_i(\varepsilon/2)$, although Lemma 3..1 ensures that the larger $\Omega_i(\varepsilon)$ suffices for the feasibility of the terminal controllers. In the analysis presented in the next section, it will be shown that tightening the terminal set in this way is required to guarantee the feasibility properties.

Before stating the CF-DMPC algorithm, an assumption is made to facilitate the initialization phase.

Assumption 3 *At initial time k_0, there exists a feasible control $\mathbf{u}_{i,k_0+l} \in \mathcal{U}_i$, $l \in \{1,\ldots,N\}$, for each $\mathcal{S}_i$, such that the solution to the full system $\mathbf{x}_{l+1+k_0} = \mathbf{A}\mathbf{x}_{l+k_0} + \mathbf{B}\mathbf{u}_{l+k_0}$, denoted $\hat{\mathbf{x}}_{\cdot|k_0,i}$, satisfies $\hat{\mathbf{x}}_{N+k_0|k_0,i} \in \Omega(\alpha\varepsilon)$ and results in a bounded cost $\bar{J}_{i,k_0}$.*

Remark 3..1 *Assumption 3 bypasses the difficult task of actually constructing an initially feasible solution in a distributed way. In fact, finding an initially feasible solution for many optimization problems is often a primary obstacle, whether or not such problems are used in a control setting. As such, many centralized implementations of MPC also assume that an initially feasible solution is available [23, 20]. Recent methods for characterizing the set of initially feasible solutions are presented in [22]. One possible way to obtain an initially feasible solution can be to solve the corresponding centralized MPC solution at the initial time instant.*

The dual-mode FC-MPC law for any $\mathcal{S}_i$, which communicates once every update, is as follows.

Algorithm 1 (CF-DMPC Algorithm)

Step 1: Initialization at time k_0.

- *Initialize $\mathbf{x}_{k_0}$, $u_{i,k_0+l-1|k_0}$, $l = 1, 2, \ldots, N$. to satisfy assumption 3*
- *At time k_0, if $\mathbf{x}_{k_0} \in \Omega(\varepsilon)$, then apply the terminal controller $\mathbf{u}_{i,k} = \mathbf{K}_i\mathbf{x}_{i,k}$, for all $k \geq k_0$; Else*
- *Compute $\hat{\mathbf{x}}_{i,k_0+l+1|k_0+1}$ according to (12) and transmit $\hat{\mathbf{x}}_{i,k_0+l+1|k_0+1}$ to $\mathcal{S}_j$, $J \in \mathcal{P}_{-i}$;*

Step 2: Communicating at time k, $k > k_0$.

- *Measure $\mathbf{x}_{i,k}$, transmit $\mathbf{x}_{i,k}$, $\hat{\mathbf{x}}_{i,k+l+1|k}$, to $\mathcal{S}_j$, $j \in \mathcal{P}_{-i}$, and receive $\mathbf{x}_{j,k}$, $\hat{\mathbf{x}}_{j,k}$ from $\mathcal{S}_j$, $j \in \mathcal{P}_{+i}$;*

Step 3: Update control law at time k.

- *If $\mathbf{x}_k \in \Omega(\varepsilon)$, then apply the terminal controller $\mathbf{u}_{i,k} = \mathbf{K}_i\mathbf{x}_{i,k}$; Else*
- *Solve Problem 1 for $\mathbf{u}_{i,k+l-1|k}$ and apply $\mathbf{u}_{i,k|k}$ to $\mathcal{S}_i$;*
- *Compute $\hat{\mathbf{x}}_{i,k+l+1|k+1}$ according to (12) and transmit $\hat{\mathbf{x}}_{i,k+l+1|k+1}$ to $\mathcal{S}_j$, $J \in \mathcal{P}_{-i}$;*

Step 4: Update control at time $k+1$.

- *Let $k+1 \to k$, repeat Step 2.*

Algorithm 1 presumes that all local controllers $\mathcal{C}_i$, $i \in \mathcal{P}$, have access to the full state $\mathbf{x}_k$. This requirement results solely from the use of the dual-mode control, in which the switching occurs synchronously only when $\mathbf{x}_k \in \Omega(\varepsilon)$, with $\Omega(\varepsilon)$ being as defined in Lemma 3..1. In the next section, it is will be shown that the CF-DMPC policy drives the state $\mathbf{x}_{k+l}$ to $\Omega(\varepsilon)$ in a finite number of updates. As a result, if $\Omega_i(\varepsilon)$ is chosen sufficiently small, then MPC can be employed for all time without switching to a terminal controller, eliminating the need of the local controllers to access the full state. Of course, in this case, instead of asymptotic stability at the origin, we can only drive the state toward the small set $\Omega(\varepsilon)$.

The analysis in the next section shows that the CF-DMPC algorithm is feasible at every update and is stabilizing.

4. Analysis

In this section, feasibility is analyzed in the first subsection, followed by stability analysis in the second subsection.

4.1. Feasibility

The main result of this section is that, provided that an initially feasible solution is available and Assumption 3 holds true, for any $\mathcal{S}_i$ and at any time $k \geq 1$, $\mathbf{u}^{\mathrm{p}}_{i,\cdot|k} = \mathbf{u}^{\mathrm{f}}_{i,\cdot|k}$ is a feasible control solution to Problem 1.

To establish this feasibility result, we will show that, for any $\mathcal{S}_i$ and at any update $k \geq 1$, the control and state pair $(\mathbf{u}^{\mathrm{f}}_{i,\cdot|k}, \mathbf{x}^{\mathrm{f}}_{i,\cdot|k})$ are a feasible solution to Problem 1 which satisfy the consistency constraints (17),(18), the control constraint (20) and the terminal state constraint (21).

Fig. 2 shows the discrepancy between the presumed state sequence $\{\hat{\mathbf{x}}_{ik+1|k}, \hat{\mathbf{x}}_{i,k+2|k}, \cdots\}$ and the predicted state sequence $\{\mathbf{x}^{\mathrm{f}}_{i,k+1|k}, \mathbf{x}^{\mathrm{f}}_{i,k+2|k}, \cdots\}$, $j \in \mathcal{P}_i$, and the relationships between these sequences and the terminal sets $\Omega_j(\varepsilon)$, $\Omega_j(\varepsilon/2)$ and $\Omega_j(\varepsilon'/2)$, where $0 < \varepsilon' = (1-\kappa)\varepsilon < \varepsilon$. To ensure feasibility, parametric conditions must be established under which $\hat{\mathbf{x}}_{i,k+N|k}$ and $\mathbf{x}^{\mathrm{f}}_{i,k+N|k}$ are within the indicated ellipsoids at the time instant k and $\hat{\mathbf{x}}_{i,k+s|k}$ and $\mathbf{x}^{\mathrm{f}}_{i,k+s|k}$ are sufficiently close to each other over the entire time interval $[k+1, k+N]$.

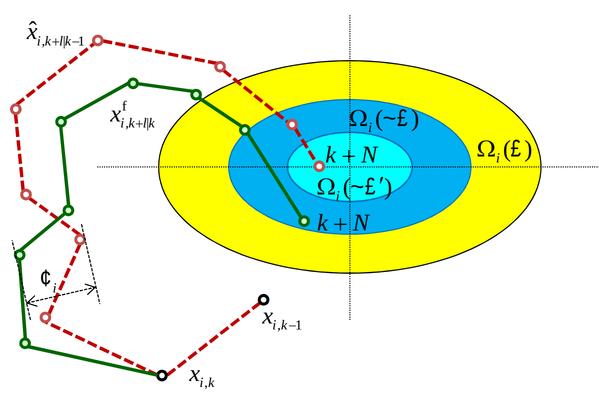

Figure 2. Schematic of the discrepancy among a feasible state sequence, the presumed state sequence and the predictive sequence.

Lemma 4..1 identifies sufficient conditions that ensure $\hat{x}_{i,k+N|k} \in \Omega_i(\varepsilon'/2)$, where $\varepsilon' = (1-\kappa)\varepsilon$. Lemma 4..2 identifies sufficient conditions that ensure $\|\mathbf{x}^{\mathrm{f}}_{i,l+k|k} - \hat{\mathbf{x}}_{i,s+k|k}\|_{\mathbf{P}_j} \leq$

$\kappa\varepsilon/(2\sqrt{m})$ for all $i \in \mathcal{P}$. Lemma 4..3 establishes that the control constraint is satisfied. Finally, Theorem 4..1 combines the results in Lemmas 4..1-4..3 to arrive at the conclusion that, for any $i \in \mathcal{P}$, the control and state pair $(\mathbf{u}^{\mathrm{f}}_{i,\cdot|k}, \mathbf{x}^{\mathrm{f}}_{i,\cdot|k})$ are a feasible solution to Problem 1 at any update $k \geq 1$.

Lemma 4..1 *Suppose that Assumptions 1-3 hold and $\mathbf{x}_{k_0} \in \mathcal{X}$. For any $k \geq 0$, if Problem 1 has a solution at time $k-1$ and $\hat{\mathbf{x}}_{i,k+N-1|k-1} \in \Omega_j(\varepsilon/2)$ for any $j \in \mathcal{P}_i, i \in \mathcal{P}$, then*

$$\hat{\mathbf{x}}_{i,k+N-1|k} \in \Omega_j(\varepsilon/2)$$

and

$$\hat{\mathbf{x}}_{i,k+N|k} \in \Omega_j(\varepsilon'/2),$$

provided that $\widehat{Q}_j$ and P_j satisfy

$$\max_{i\in\mathcal{P}}(\rho_i) \leq 1-\kappa \tag{25}$$

and $\varepsilon' = (1-\kappa)\varepsilon$,

$$\rho = \lambda_{\max}\sqrt{\left(\hat{\mathbf{Q}}_i\mathbf{P}_i^{-1}\right)^{\mathrm{T}}\hat{\mathbf{Q}}_i\mathbf{P}_i^{-1}}$$

Proof: Since Problem 1 has a solution at time $k-1$, by construction (13) and (14), it has

$$\left\|\hat{\mathbf{x}}_{i,k+N-1|k}\right\|_{\mathbf{P}_j} = \left\|\mathbf{x}^{\mathrm{p}}_{i,k+N-1|k-1}\right\|_{\mathbf{P}_i} \leq \frac{\varepsilon}{2\sqrt{m}} \tag{26}$$

In addition, since

$$\begin{aligned}\hat{\mathbf{x}}_{i,k+N|k} &= \mathbf{A}_{\mathrm{d}i}\mathbf{x}^{\mathrm{p}}_{i,k+N-1|k-1} + \sum_{j\in\mathcal{P}_{+i}}\mathbf{A}_{ij}\mathbf{x}^{\mathrm{p}}_{j,k+N-1|k-1}\\ &= \mathbf{A}_{\mathrm{d}i}\hat{\mathbf{x}}_{i,k+N-1|k} + \sum_{j\in\mathcal{P}_{+i}}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k+N-1|k}\end{aligned} \tag{27}$$

It has

$$\left\|\hat{\mathbf{x}}_{i,k+N|k}\right\|_{\mathbf{P}_i} = \left\|\mathbf{A}_{\mathrm{d}i}\hat{\mathbf{x}}_{i,k+N-1|k} + \sum_{j\in\mathcal{P}_{+i}}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k+N-1|k}\right\|_{\mathbf{P}_i} \tag{28}$$

Consider Assumption 2, $\mathbf{A}_{\mathrm{o}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{o}} + \mathbf{A}_{\mathrm{o}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{d}} + \mathbf{A}_{\mathrm{d}}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{o}} < \hat{\mathbf{Q}}/2$
Thus if , then

$$\begin{aligned}\left\|\hat{\mathbf{x}}_{i,k+N|k}\right\|_{\mathbf{P}_i} &\leq \left\|\hat{\mathbf{x}}_{i,k+N-1|k}\right\|_{\hat{\mathbf{Q}}/2}\\ &\leq \lambda_{\max}\sqrt{\left(\hat{\mathbf{Q}}_i\mathbf{P}_i^{-1}\right)^{\mathrm{T}}\hat{\mathbf{Q}}_i\mathbf{P}_i^{-1}}\left\|\hat{\mathbf{x}}_{i,k+N-1|k}\right\|_{\mathbf{P}_i}\\ &\leq (1-\kappa)\frac{\varepsilon}{2\sqrt{m}}\end{aligned} \tag{29}$$

This completes the proof of Lemma 4..1 □

Lemma 4..2 *Suppose that Assumptions 1-3 hold and $x(k_0) \in \mathcal{X}$. For any $k \geq 0$, if Problem 1 has a solution at every update time l, $l = 1, 2, \cdots, k-1$, then*

$$\left\|\mathbf{x}^{\mathrm{f}}_{i,k+l|k} - \hat{\mathbf{x}}_{i,k+l|k}\right\|_{\mathbf{P}_i} \leq \frac{\kappa\varepsilon}{2\sqrt{m}}, \tag{30}$$

for all $i \in \mathcal{P}_i$ and all $l = 1, 2, \cdots, N$, provided that (25) and the following parametric condition hold

$$\frac{\sqrt{m_2}}{\xi\lambda_{\min}(P)} \sum_{l=0}^{N-2} \alpha_l \leq 1, \tag{31}$$

where α_l is as defined in (23). Furthermore, the feasible control $\mathbf{u}^{\mathrm{f}}_{i,k+s|k}$ and the feasible state $\mathbf{x}^{\mathrm{f}}_{i,k+l|k}$ satisfy constraints (17)-(18).

Proof: We will prove (30) first. Since a solution exists at update time $1, 2, \ldots, k-1$, according to (12), (13) and (24), for any $s = 1, 2, \cdots, N-1$, the feasible state is given by

$$\begin{aligned}
\mathbf{x}^{\mathrm{f}}_{i,k+l|k} &= \mathbf{A}^{l}_{ii}\mathbf{x}^{\mathrm{f}}_{i,k|k} + \sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{B}_{ii}\mathbf{u}^{\mathrm{f}}_{i,k+l|k} + \sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k+h-1|k} \\
&= \mathbf{A}^{l}_{ii}\left(\mathbf{A}^{l}_{ii}\mathbf{x}_{i,k-1|k-1} + \mathbf{B}_{ii}\mathbf{u}_{i,k-1|k-1} + \sum_{j\in\mathcal{P}_{+i}}\mathbf{A}_{ij}\mathbf{x}_{j,k-1|k-1}\right) \\
&\quad + \sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{B}_{ii}\hat{\mathbf{u}}_{i,k+l|k} + \sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{A}_{ij}\mathbf{x}^{\mathrm{p}}_{j,k+h-1|k-1}
\end{aligned} \tag{32}$$

and the presumed state is

$$\begin{aligned}
\hat{\mathbf{x}}_{i,k+l|k} &= \mathbf{A}^{l}_{ii}\mathbf{x}_{i,k|k-1} + \sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{B}_{ii}\mathbf{u}_{i,k+l|k-1} + \sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k+h-1|k-1} \\
&= \mathbf{A}^{l}_{ii}\left(\mathbf{A}^{l}_{ii}\mathbf{x}_{i,k-1|k-1} + \mathbf{B}_{ii}\mathbf{u}_{i,k-1|k-1} + \sum_{j\in\mathcal{P}_{+i}}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k-1|k-1}\right) \\
&\quad + \sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{B}_{ii}\hat{\mathbf{u}}_{i,k+l|k} + \sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{A}_{ij}\hat{\mathbf{x}}_{j,k+h-1|k-1}
\end{aligned} \tag{33}$$

Subtracting (34) from (33), and from the definition of (23) we obtain the discrepancy between the feasible state sequence and the presumed state sequence as

$$\begin{aligned}
\left\|\mathbf{x}^{\mathrm{f}}_{i,k+l|k} - \hat{\mathbf{x}}_{i,k+l|k}\right\|_{\mathbf{P}_i} &= \left\|\sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{A}_{ij}\left(\mathbf{x}^{\mathrm{p}}_{i,k+h-1|k-1} - \hat{\mathbf{x}}_{j,k+h-1|k-1}\right)\right\|_{\mathbf{P}_i} \\
&\leq \sum_{j\in\mathcal{P}_{+i}}\sum_{h=1}^{l}\mathbf{A}^{l-h}_{ii}\mathbf{A}_{ij}\left\|\mathbf{x}^{\mathrm{p}}_{i,k+h-1|k-1} - \hat{\mathbf{x}}_{j,k+h-1|k-1}\right\|_{\mathbf{P}_i} \\
&\leq \sum_{s=1}^{l}\alpha_{l-s}\left\|\mathbf{x}^{\mathrm{p}}_{i,k+s-1|k-1} - \hat{\mathbf{x}}_{i,k+s-1|k-1}\right\|_2.
\end{aligned} \tag{34}$$

Let $\mathcal{S}_r$ be the subsystem which maximizes

$$\sum_{h=1}^{l} \alpha_{l-h} \left\| \mathbf{x}^{\mathrm{p}}_{i,k-1+h|k-1} - \hat{\mathbf{x}}_{i,k-1+h|k-1} \right\|_2, i \in \mathcal{P} \tag{35}$$

Then, the following equation can be deduced from (35)

$$\left\| \mathbf{x}^{f}_{j,k+l|k} - \hat{\mathbf{x}}_{j,k+l|k} \right\|_{\mathbf{P}_i} \leq \sqrt{m_1} \sum_{h=1}^{l} \alpha_{l-h} \left\| \mathbf{x}^{\mathrm{p}}_{g,k+h-1|k-1} - \hat{\mathbf{x}}_{g,k+h-1|k-1} \right\|_2 \tag{36}$$

Since $\mathbf{x}^{\mathrm{p}}_{i,l|k-1}$ satisfy constraints (17) for all times $l = 1, 2, \ldots, k-1$, the following equation can be deduced

$$\begin{aligned} & \left\| \mathbf{x}^{\mathrm{f}}_{i,k+l|k} - \hat{\mathbf{x}}_{i,k+l|k} \right\|_{\mathbf{P}_i} \\ \leq \ & \frac{(1-\xi)(1-\kappa)\varepsilon}{2\sqrt{m}} + \frac{\xi(1-\kappa)\varepsilon}{2\sqrt{m}} \\ = \ & \frac{\kappa\varepsilon}{2\sqrt{m}}. \end{aligned} \tag{37}$$

Thus, (30) holds for all $l = 1, 2, \ldots, N-1$.

When $l = N$, it has

$$\mathbf{x}^{\mathrm{f}}_{i,k+N|k} = \mathbf{A}_{\mathrm{d},i} \mathbf{x}^{\mathrm{f}}_{i,k+N|k} + \sum_{j \in \mathcal{P}_{+i}} \mathbf{A}_{ij} \hat{\mathbf{x}}_{j,k+N-1|k} \tag{38}$$

$$\hat{\mathbf{x}}_{i,k+N|k} = \mathbf{A}_{\mathrm{d},i} \hat{\mathbf{x}}_{i,k+N-1|k} + \sum_{j \in \mathcal{P}_{+i}} \mathbf{A}_{ij} \hat{\mathbf{x}}_{j,k+N-1|k} P \tag{39}$$

From the subtraction of the two equations, then, the discrepancy between the feasible state $\mathbf{x}^{\mathrm{f}}_{i,(k+N)}$ and the presumed state $\hat{\mathbf{x}}_{i,(k+N)}$ is

$$\mathbf{x}^{\mathrm{f}}_{i,k+N|k} - \hat{\mathbf{x}}_{i,k+N|k} = \mathbf{A}_{\mathrm{d},i} \left(\mathbf{x}^{\mathrm{f}}_{i,k+N-1|k} - \hat{\mathbf{x}}_{i,k+N-1|k} \right) \tag{40}$$

Consequently, (30) holds for all $l = 1, 2, \ldots, N$.

In what follows we will prove that the feasible state $\mathbf{x}^{\mathrm{f}}_{i,(k+l)}$ satisfy constraints (17)(18) when (30) holds.

When $l = 1, 2, \ldots, N-1$, substitute $\mathbf{x}^{\mathrm{f}}_{i,k+l|k}$ in the constraint (17) with considering (31), we can get

$$\begin{aligned} & \sum_{h=1}^{l} \alpha_{l-h} \left\| \mathbf{x}^{\mathrm{f}}_{i,k+h|k} - \hat{\mathbf{x}}_{i,k+h|k} \right\|_2 \\ \leq \ & \frac{1}{\lambda_{\min}(\mathbf{P}_i)} \sum_{l=1}^{s} \alpha_{l-h} \left\| \mathbf{x}^{\mathrm{f}}_{i,k+h|k} - \hat{\mathbf{x}}_{i,k+h|k} \right\|_{\mathbf{P}_i} \\ \leq \ & \frac{1}{\lambda_{\min}(\mathbf{P})} \sum_{h=1}^{l} \alpha_{l-h} \frac{\sqrt{m_1}}{\xi} \frac{\xi\kappa\varepsilon}{2\sqrt{m m_1}} \end{aligned} \tag{41}$$

Thus, when

$$\frac{\sqrt{m_1}}{\xi\lambda_{\min}(\mathbf{P})}\sum_{h=1}^{l}\alpha_{l-h}\leq 1,$$

state $\mathbf{x}^{\mathrm{f}}_{i,k+l|k}$, $l=1,2,\ldots,N-1$, satisfy constraint constraint (17).

Finally, when $l=N$,

$$\left\|\mathbf{x}^{\mathrm{f}}_{i,k+N|k}-\hat{\mathbf{x}}_{i,k+N|k}\right\|_{\mathbf{P}_i}\leq\frac{\kappa\varepsilon}{2\sqrt{m}},\tag{42}$$

which shows constraint (18) is satisfied. The proof is completed. □

In what follows we establish that, at time k, if the conditions (25),(30) and (31) are satisfied, then $\mathbf{u}^{\mathrm{f}}_{i,k+l-1|k}$, $l=1,2,\ldots,N$, is feasible solution of Problem 1.

Lemma 4..3 *Suppose that Assumptions 1-3 hold, $\mathbf{x}_{k_0}\in\mathbf{R}^{n_x}$, and conditions (25) and (31) are satisfied. For any $k\geq 0$, if Problem 1 has a solution at every update time t, $t=1,2,\cdots,k-1$, then $\mathbf{u}^{\mathrm{f}}_{i,k+l|k}\in\mathcal{U}$ for all $l=1,2,\cdots,N-1$.*

Proof: Since Problem 1 has a feasible solution at $t=1,2,\cdots,k-1$, and $\mathbf{u}^{\mathrm{f}}_{i,k+s-1|k}=\mathbf{u}^{\mathrm{p}}_{i,k+s-1|k-1}$ for all $l=1,2,\cdots,N-1$, we only need to show that $\mathbf{u}^{\mathrm{f}}_{i,k+N-1|k}\in\mathcal{U}$.

Since ε has been chosen to satisfy the conditions of Lemma 3..1, $\mathbf{K}_i\mathbf{x}_i\in\mathcal{U}$ for all $i\in\mathcal{P}$ when $\mathbf{x}\in\Omega(\varepsilon)$. Consequently, a sufficient condition for $\mathbf{u}^{\mathrm{f}}_{i,k+N-1|k}\in\mathcal{U}$ is that $\mathbf{x}^{\mathrm{f}}_{i,k+N-1|k}\in\Omega(\varepsilon)$.

In view of Lemmas 4..1 and 4..2, using the triangle inequality, we have

$$\begin{aligned}
&\|\mathbf{x}^{\mathrm{f}}_{i,k+N-1|k}\|_{\mathbf{P}_i}\\
\leq\;&\left\|\mathbf{x}^{\mathrm{f}}_{i,k+N-1|k}-\hat{\mathbf{x}}_{i,k+N-1|k}\right\|_{\mathbf{P}_i}\\
&+\left\|\hat{\mathbf{x}}_{i,k+N-1|k}\right\|_{\mathbf{P}_i}\\
\leq\;&\frac{\varepsilon}{2(q+1)\sqrt{m}}+\frac{\varepsilon}{2\sqrt{m}}\\
\leq\;&\frac{\varepsilon}{\sqrt{m}},
\end{aligned}$$

that is, $\mathbf{x}^{\mathrm{f}}_{i,k+N|k}\in\Omega_i(\varepsilon)$. This concludes the proof. □

Lemma 4..4 *Suppose that Assumptions 1 and 3 hold, $\mathbf{x}_{k_0}\in\mathcal{X}$, and the conditions (30) and (31) are satisfied. For any $k\geq 0$, if Problem 1 has a solution at every update time t, $t=0,\ldots,k-1$, then the terminal state constraint $\mathbf{x}^{\mathrm{f}}_{i,k+N|k}\in\Omega\left(\varepsilon/2\right)$ is hold, for any $i\in\mathcal{P}$.*

Proof: Since there is a solution for Problem 1 at updates $t=1,\ldots,k-1$, Lemma 4..1-4..3

can be invoked. Using the triangle inequality, it has

$$\begin{aligned}
&\|\mathbf{x}^{\mathrm{f}}_{i,k+N|k}\|_{\mathbf{P}_i}\\
\leq\ &\|\mathbf{x}^{\mathrm{f}}_{i,k+N|k}-\hat{\mathbf{x}}_{i,k+N|k-1}\|_{\mathbf{P}_i}+\|\hat{\mathbf{x}}_{i,k+N|k-1,i}\|_{\mathbf{P}_i}\\
\leq\ &\frac{\kappa\varepsilon}{2\sqrt{m}}+\frac{(1-\kappa)\varepsilon}{2\sqrt{m}}\\
=\ &\frac{\varepsilon}{2\sqrt{m}}
\end{aligned}\tag{43}$$

for each $i\in\mathcal{P}$. This shows that the terminal state constraint is satisfied. This completes the proof. $\square$

Theorem 4..1 *Suppose that Assumptions 1-3 hold,* $\mathbf{x}(k_0)\in\mathbf{R}^{n_x}$ *and constraints (17),(18) and (20) are satisfied at* k_0*. Then, for every* $i\in\mathcal{P}$*, the control* $\mathbf{u}^{\mathrm{f}}_{i,\cdot|k}$ *and state* $\mathbf{x}^{\mathrm{f}}_{i,\cdot|k}$*, respectively defined by* (24)*, are a feasible solution of Problem 1 at every update* $k\geq 1$.

Proof: We will prove the theorem by induction.

First, consider the case of $k=1$. The state sequence $\mathbf{x}^{\mathrm{p}}_{i,\cdot|1}=\mathbf{x}^{\mathrm{f}}_{i,\cdot|1}$ trivially satisfies the dynamic equation (12), the stability constraint (19) and the consistency constraints (17)-(18).

Observe that

$$\hat{\mathbf{x}}_{i,1|1}=\mathbf{x}^{\mathrm{p}}_{i,1|0}=\mathbf{x}^{\mathrm{f}}_{i,1|1}=\mathbf{x}_{i,1},\ i\in\mathcal{P},$$

and that

$$\mathbf{x}^{\mathrm{f}}_{i,1+l|1}=\mathbf{x}^{\mathrm{p}}_{i,1+l|0},\ l=1,2,\cdots,N-1.$$

Thus, $x^{\mathrm{f}}_{i,N|1}\in\Omega_i(\varepsilon/2)$. By the invariance of $\Omega(\varepsilon)$ under the terminal controller and the conditions in Lemma 3..1, it follows that the terminal state and control constraints are also satisfied. This completes the proof of the case of $k=1$.

Now suppose that $\mathbf{u}^{\mathrm{p}}_{i,\cdot|l}=\mathbf{u}^{\mathrm{f}}_{i,\cdot|l}$ is a feasible solution for $l=1,2,\cdots,k-1$. We will show that $\mathbf{u}^{\mathrm{f}}_{i,\cdot|k}$ is a feasible solution at update k.

As before, the consistency constraint (17) is trivially satisfied, and $\mathbf{u}^{\mathrm{f}}_{i,\cdot|k}$ is the corresponding state sequence that satisfies the dynamic equation. Since there is a solution for Problem 1 at updates $l=1,2,\cdots,k-1$, Lemmas 4..1-4..3 can be invoked. Lemma 4..3 guarantees control constraint feasibility. Lemma 4..4 shows that the terminal state constraint is satisfied and the proof of Theorem 4..1 is completed. $\square$

4.2. Stability

The stability of the closed-loop system is analyzed in this subsection.

Theorem 4..2 *Suppose that Assumptions 1-3 hold,* $x(k_0)\in\mathbf{R}^{n_x}$*, constraints (17)-(19) and (20) are satisfied, and the following parametric condition holds*

$$\kappa\frac{N-1}{2}+\frac{1}{\mu}<\frac{1}{2}.\tag{44}$$

Then, by application of Algorithm 1, the closed-loop system (2) is asymptotically stable at the origin.

Proof: By Algorithm 1 and Lemma 3..1, when $\mathbf{x}(k)$ enters $\Omega(\varepsilon)$, the terminal controllers take over to keep it in there and stabilize the system at the origin. Therefore, it remains to show that if $\mathbf{x}(0) \in \mathcal{X} \setminus \Omega(\varepsilon)$, then by the application of Algorithm 1, the state of system (2) is driven to the set $\Omega(\varepsilon)$ in finite time.

Define the non-negative function for $\mathcal{S}$

$$V_k = \sum\nolimits_{l=1}^{N} \|\mathbf{x}^{\mathrm{p}}_{k+l|k}\|_{\mathbf{P}}.$$

In what follows, we will show that, for any $k \geq 0$, if $\mathbf{x}_k \in \mathcal{X} \setminus \Omega(\varepsilon)$, then there exists a constant $\eta \in (0, \infty)$ such that $V_k \leq V_{k-1} - \eta$. Constraint (19) implies that

$$\left\|\mathbf{x}^{\mathrm{p}}_{k+l|k}\right\|_{\mathbf{P}} \leq \left\|\mathbf{x}^{\mathrm{f}}_{k+l|k}\right\|_{\mathbf{P}} + \frac{\varepsilon}{\mu N}.$$

Therefore,

$$V_k \leq \sum\nolimits_{l=1}^{N} \left\|\mathbf{x}^{\mathrm{f}}_{k+l|k}\right\|_{\mathbf{P}} + \frac{\varepsilon}{\mu}.$$

Subtracting V_{k-1} from V_k and using $\mathbf{x}^{\mathrm{p}}_{k+l|k-1} = \hat{\mathbf{x}}_{k+l|k}$, $l = 1, 2, \cdots, N-1$, gives

$$\begin{aligned} & V_k - V_{k-1} \\ \leq \; & -\left\|\mathbf{x}^{\mathrm{p}}_{k|k-1}\right\|_{\mathbf{P}} + \frac{\varepsilon}{\mu} + \left\|\mathbf{x}^{\mathrm{f}}_{k+N|k}\right\|_{\mathbf{P}} \\ & + \sum_{l=1}^{N-1} \left(\left\|\mathbf{x}^{\mathrm{f}}_{k+l|k}\right\|_{\mathbf{P}} - \|\hat{\mathbf{x}}_{k+l|k}\|_{\mathbf{P}} \right). \end{aligned} \tag{45}$$

Assuming $\mathbf{x}_k \in \mathcal{X} \setminus \Omega(\varepsilon)$ yields

$$\left\|\mathbf{x}^{\mathrm{p}}_{k|k-1}\right\|_{\mathbf{P}} > \varepsilon. \tag{46}$$

Also, by Theorem 4..1 we have

$$\left\|\mathbf{x}^{\mathrm{f}}_{k+N|k)}\right\|_{\mathbf{P}} \leq \varepsilon/2, \tag{47}$$

and by Lemma 4..2, we have

$$\sum_{l=1}^{N-1} \left(\left\|\mathbf{x}^{\mathrm{f}}(k+l|k)\right\|_{\mathbf{P}} - \|\hat{\mathbf{x}}_{k+l|k}\|_{\mathbf{P}} \right) \leq \frac{(N-1)\kappa\varepsilon}{2}. \tag{48}$$

Using (46)-(48) in (45) then yields

$$V_k - V_{k-1} < \varepsilon \left(-1 + \frac{(N-1)\kappa}{2} + \frac{1}{2} + \frac{1}{\mu} \right), \tag{49}$$

which, in view of (44), implies that $V_k - V_{k-1} < 0$. Thus, for any $k \geq 0$, if $\mathbf{x}_k \in \mathcal{X} \setminus \Omega(\varepsilon)$, there is a constant $\eta \in (0, \infty)$ such that $V_k \leq V_{k-1} - \eta$. It then follows that there exists a finite time k' such that $\mathbf{x}_{k'} \in \Omega(\varepsilon)$. This concludes the proof. □

We have now established the feasibility the CF-DMPC and the stability of the resulting closed-loop system. That is, if an initially feasible solution could be found, subsequent feasibility of the algorithm is guaranteed at every update, and the resulting closed-loop system is asymptotically stable at the origin.

5. Simulation

The multi-zones building temperature regulation systems are a class of typical spatially distributed systems, as shown in Fig. 3, which are composed of many physically interacted subsystems (rooms or zones) labeled with $\mathcal{S}_1, \mathcal{S}_2, \ldots$, respectively. The thermal influences between rooms of the same building occur through internal walls (the internal walls isolation is weak) and/or door openings. A thermal-meter and a heater (or air condition) are installed in each zone, which is used to measure and adjust the temperature of the multi-zones building.

For simplicity, the 7-zones building is taken as example. The relationship among these seven zones is also shown in Fig. 3., where zone $\mathcal{S}_1$ is impacted by zone $\mathcal{S}_2$ and zone $\mathcal{S}_7$; zone $\mathcal{S}_2$ is impacted by zone $\mathcal{S}_1$, $\mathcal{S}_3$ and zone $\mathcal{S}_7$; zone $\mathcal{S}_3$ is impacted by zone $\mathcal{S}_2$, $\mathcal{S}_4$ and zone $\mathcal{S}_7$; zone $\mathcal{S}_4$ is impacted by zone $\mathcal{S}_3$, $\mathcal{S}_5$ and zone $\mathcal{S}_7$; zone $\mathcal{S}_5$ is impacted by zone $\mathcal{S}_4$, $\mathcal{S}_6$ and zone $\mathcal{S}_7$; zone $\mathcal{S}_6$ is impacted by zone $\mathcal{S}_5$ and zone $\mathcal{S}_7$; zone $\mathcal{S}_4$ is impacted by all the other zones.

Let $\mathcal{U}_i$ be defined to reflect both the constraint on the input $u_i \in [u_{i,\mathrm{L}}, u_{i,\mathrm{U}}]$ and the constraint on the increment of the input $\Delta u_i \in [\Delta u_{i,\mathrm{L}}, \Delta u_{i,\mathrm{U}}]$. The models of these seven subsystems are respectively given by

$$
\begin{aligned}
x_{1,k+1} &= 0.574x_{1,k} + 0.384u_{1,k} + 0.029x_{2,k} + 0.057x_{7,k} \\
x_{2,k+1} &= 0.535x_{2,k} + 0.372u_{2,k} + 0.054x_{1,k} + 0.054x_{3,k} \\
&\quad +0.054x_{7,k} \\
x_{3,k+1} &= 0.547x_{3,k} + 0.376u_{3,k} \\
&\quad +0.055x_{2,k} + 0.055x_{4,k} + 0.055x_{7,k} \\
x_{4,k+1} &= 0.606x_{4,k} + 0.394u_{4,k} \\
&\quad +0.061x_{3,k} + 0.061x_{5,k} + 0.061x_{7,k}, \\
x_{5,k+1} &= 0.681x_{5,k} + 0.415u_{5,k} \\
&\quad +0.068x_{4,k} + 0.068x_{6,k} + 0.068x_{7,k} \\
x_{6,k+1} &= 0.548x_{6,k} + 0.376u_{6,k} \\
&\quad +0.055x_{5,k} + 0.055x_{7,k}, \\
x_{7,k+1} &= 0.716x_{7,k} + 0.425u_{7,k} + 0.018x_{1,k} + 0.018x_{2,k} \\
&\quad +0.018x_{3,k} + 0.018x_{4,k} + 0.018x_{5,k} + 0.018x_{6,k}.
\end{aligned}
$$

For the purpose of comparison, the centralized MPC, the DMPC where each subsystem-based MPC optimizes its own local cost [10], here we call it as local cost optimization based DMPC (LCO-DMPC), and the propsed CF-DMPC are all applied to this system.

Some parameters of the controllers in proposed CF-DMPC are shown in Table 2. Among these parameters, P_i is obtained by solving the Lyapnov function. The eigenvalue of each closed-loop system under the feedback control shown in the table is 0.5. Set $\varepsilon = 0.15$, and set the control horizon of all the controllers to be $N = 10$. Set the initial presumed inputs and states, at time $k_0 = 0$, be zeros.

In both the centralized MPC and the subsystem-based MPCs of the local cost optimization based DMPC, the dual mode strategy is adopted, and set the parameters, the initial states and the initial presumed inputs be the same as those used in CF-DMPC.

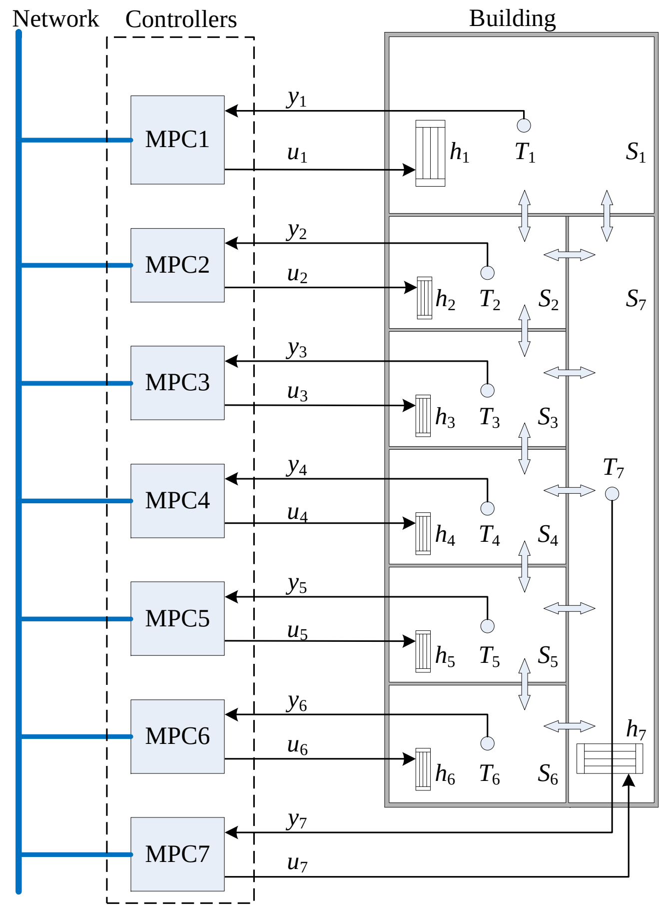

Figure 3. The interaction relationship among subsystems.

Table 2. Parameters of CF-DMPC

$Sub-system$	K_i	P_i	Q_i	R_i	$\Delta u_{i,\mathrm{U}}$ $\Delta u_{i,\mathrm{L}}$
$\mathcal{S}_1$	-0.44	5.38	4	0.2	± 1
$\mathcal{S}_2$	-0.34	5.36	4	0.2	± 1
$\mathcal{S}_3$	-0.37	5.37	4	0.2	± 1
$\mathcal{S}_4$	-0.52	5.40	4	0.2	± 1
$\mathcal{S}_5$	-0.68	5.46	4	0.2	± 1
$\mathcal{S}_6$	-0.37	5.37	4	0.2	± 1
$\mathcal{S}_7$	-0.76	5.49	4	0.2	± 1

The state responses and the inputs of the closed-loop system under the control of the centralized MPC, CF-DMPC and LCO-DMPC are shown in Figs. 4. The shape of the s-tate response curves under the control CF-DMPC are similar to those under the centralized MPC. Under the CF-DMPC control design, when set point changed, there is no significant overshooting, but some fluctuations exist in the trajectories of states of the interacting sub-

Table 3. State square errors of the closed-loop system under the control of the centralized MPC (CMPC), the LCO-DMPC and the CF-DMPC

Items	CMPC	CF-DMPC	LCO-DMPC
$\mathcal{S}_1$	0.0109	0.1146	2.0891
$\mathcal{S}_2$	2.2038	3.0245	6.2892
$\mathcal{S}_3$	5.4350	6.9908	10.6391
$\mathcal{S}_4$	2.2480	3.2122	15.3015
$\mathcal{S}_5$	4.5307	5.6741	30.2392
$\mathcal{S}_6$	4.3403	5.4926	8.2768
$\mathcal{S}_7$	9.2132	11.0574	33.6902
Total	27.9819	35.5663	106.5251

systems. Under the LCO-DMPC control design, the states of all subsystems could converge to set point, but there exists much larger overshooting comparing to those under the control of FC-DMPC and centralized MPC, and there are larger amplitude in the fluctuating of state than those under the control of CF-DMPC.

Table 3 shows the state square errors of the closed-loop system under the control of the centralized MPC, the CF-DMPC and the local cost optimization based DMPC, respectively. The total errors under the CF-DMPC is 7.5844 (27.1%) larger than that under the centralized MPC. The total errors resulting from the LCO-DMPC is 78.5432 (280.7%) larger than that results from the centralized MPC. The performance of the CF-DMPC is significantly better than that of the LCO-DMPC.

Table 4 shows the required network connectivity under the control of the centralized MPC, the CF-DMPC and the LCO-DMPC, respectively. The required network connectivity under the control of CF-DMPC equals to that under the control of LCO-DMPC and is much less than that under the control of centralized MPC.

Table 4. Required network connectivity under the control of the centralized MPC (CMPC), the LCO-DMPC and the CF-DMPC

Items	CMPC	CF-DMPC	LCO-DMPC
$\mathcal{S}_1$	All	2,7	2,7
$\mathcal{S}_2$	All	1,3,7	1, 3,7
$\mathcal{S}_3$	All	2, 4,7	2, 4,7
$\mathcal{S}_4$	All	3,5,7	3,5,7
$\mathcal{S}_5$	All	4,6,7	4,6,7
$\mathcal{S}_6$	All	5,7	5,7
$\mathcal{S}_7$	All	All	All

From these simulation results, it can be seen that the proposed CF-DMPC is able to steer the system states to the set point if there is a feasible solution at the initial states, and it could obtain a better global performance than LCO-DMPC when the same network connectivity provided. The global performance of entire closed-loop system is improved without any weakening of the characteristics of good error tolerance and high flexibility of

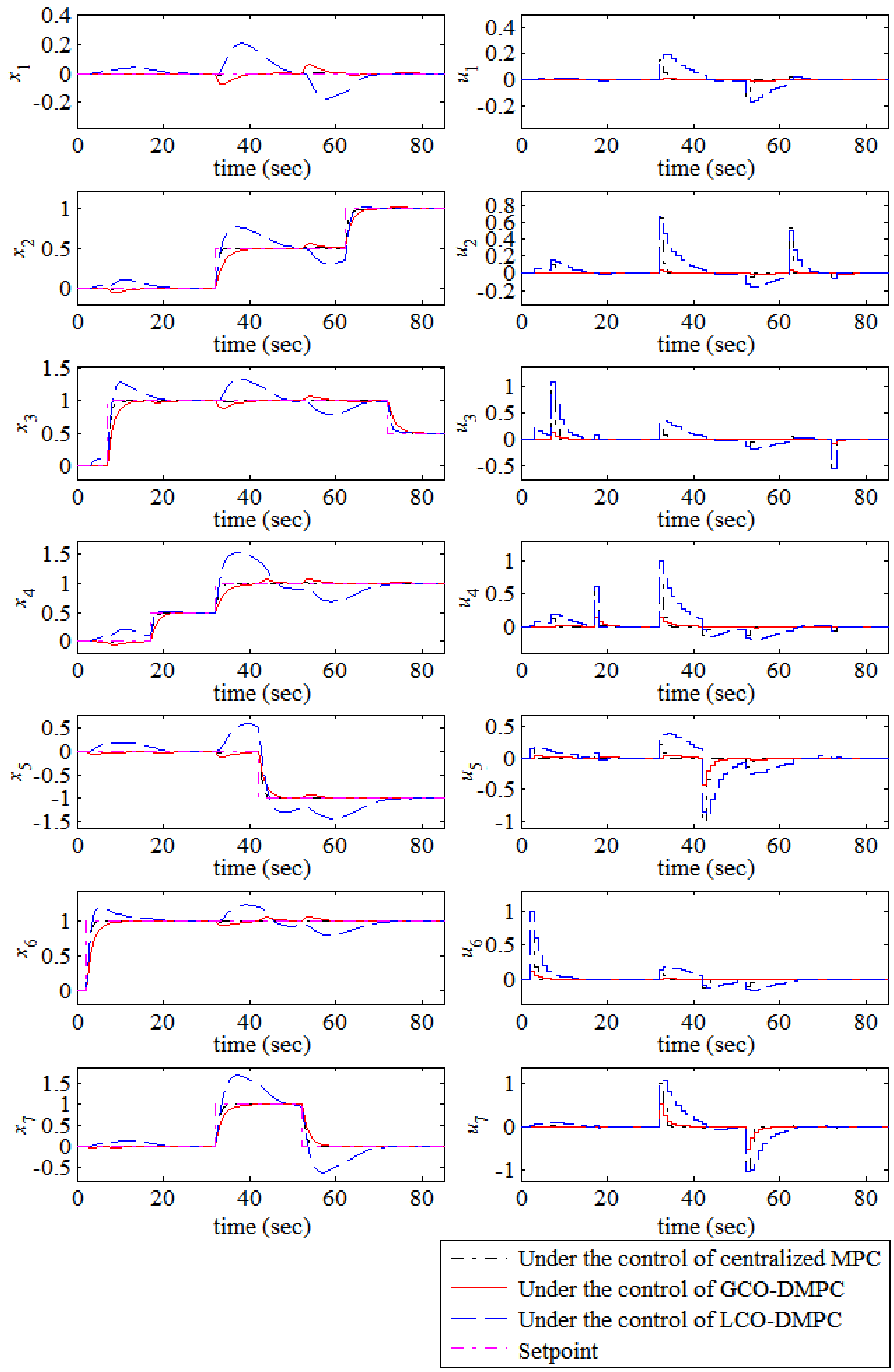

Figure 4. The evolution of the states under the centralized MPC, LCO-DMPC and CF-DMPC.

the whole control system.

6. Conclusion

In this paper, a Coordinated Flexible DMPC proposed for distributed system is developed for dynamically coupled spatially distributed systems subject to decoupled input constraints. The proposed method could improve the global performance of entire closed-loop system without any increasing of network connectivity. In addition, if an initially feasible solution and a feed back control law $K_i x$ could be found, the subsequent feasibility of the algorithm is guaranteed at every update, and the resulting closed-loop system is asymptotically stable. The simulations illustrate that the performance of global system under the control of proposed method is very close to that under the control of centralized MPC.

7. Acknowledgement

This work was supported by the National Nature Science Foundation of China (61233004, 61221003, 61374109, 61304078,61473187), the National Basic Research Program of China (973 Program-2013CB035500), and partly sponsored by the International Cooperation Program of Shanghai Science and Technology Commission (12230709600), the Higher Education Research Fund for the Doctoral Program of China (20120073130006 & 20110073110018), and the China Postdoctoral Science Foundation (2013M540364).

References

[1] Walid Al-Gherwi, Hector Budman, and Ali Elkamel. Selection of control structure for distributed model predictive control in the presence of model errors. *Journal of Process Control*, 20(3):270–284, 2010.

[2] Walid Al-Gherwi, Hector Budman, and Ali Elkamel. A robust distributed model predictive control algorithm. *Journal of Process Control*, 21(8):1127–1137, 2011.

[3] I Alvarado, D Limon, D Muñoz de la Peña, JM Maestre, MA Ridao, H Scheu, W Marquardt, RR Negenborn, B De Schutter, F Valencia, et al. A comparative analysis of distributed mpc techniques applied to the hd-mpc four-tank benchmark. *Journal of Process Control*, 21(5):800–815, 2011.

[4] E. Camponogara, D. Jia, B.H. Krogh, and S. Talukdar. Distributed model predictive control. *IEEE Control Systems Magazine*, 22(1):44–52, 2002.

[5] Eduardo Camponogara and Helton F Scherer. Distributed optimization for model predictive control of linear dynamic networks with control-input and output constraints. *IEEE Transactions on Automation Science and Engineering*, 8(1):233–242, 2011.

[6] Ruoyu Cheng, J Fraser Forbes, and W San Yip. Dantzig–wolfe decomposition and plant-wide mpc coordination. *Computers & Chemical Engineering*, 32(7):1507–1522, 2008.

[7] Panagiotis D Christofides, Riccardo Scattolini, David Muñoz de la Peña, and Jinfeng Liu. Distributed model predictive control: A tutorial review and future research directions. *Computers & Chemical Engineering*, 51:21–41, 2013.

[8] Alejandro J del Real, Alicia Arce, and Carlos Bordons. Combined environmental and economic dispatch of smart grids using distributed model predictive control. *International Journal of Electrical Power & Energy Systems*, 54:65–76, 2014.

[9] X. Du, Y. Xi, and S. Li. Distributed model predictive control for large-scale systems. In *American Control Conference, 2001. Proceedings of the 2001*, volume 4, pages 3142–3143. IEEE, 2001.

[10] W.B. Dunbar. Distributed receding horizon control of dynamically coupled nonlinear systems. *IEEE Transactions on Automatic Control*, 52(7):1249–1263, 2007.

[11] Marcello Farina and Riccardo Scattolini. Distributed predictive control: a non-cooperative algorithm with neighbor-to-neighbor communication for linear systems. *Automatica*, 48(6):1088–1096, 2012.

[12] V Kirubakaran, TK Radhakrishnan, and N Sivakumaran. Distributed multiparametric model predictive control design for a quadruple tank process. *Measurement*, 47:841–854, 2014.

[13] J.W. Lee, W. Hyun Kwon, and J. Choi. On stability of constrained receding horizon control with finite terminal weighting matrix. *Automatica*, 34(12):1607–1612, 1998.

[14] S. Leirens, C. Zamora, RR Negenborn, and B. De Schutter. Coordination in urban water supply networks using distributed model predictive control. In *American Control Conference (ACC), 2010*, pages 3957–3962. IEEE, 2010.

[15] S. Li, Y. Zhang, and Q. Zhu. Nash-optimization enhanced distributed model predictive control applied to the shell benchmark problem. *Information Sciences*, 170(2-4):329–349, 2005.

[16] S. Li, Y. Zheng, and Z. Lin. Impacted-region optimization for distributed model predictive control systems with constraints. *Automation Science and Engineering, IEEE Transactions on*, PP(99):1–14, 2014.

[17] S. Li, Y. Zheng, and B. Wang. Steady-state target calculation for constrained predictive control systems based on goal programming. *Asia-Pacific Journal of Chemical Engineering*, 3(6):648–655, 2008.

[18] Jinfeng Liu, David Muñoz de la Peña, and Panagiotis D Christofides. Distributed model predictive control of nonlinear systems subject to asynchronous and delayed measurements. *Automatica*, 46(1):52–61, 2010.

[19] J.M. Maciejowski. *Predictive control: with constraints*. Pearson education, 2002.

[20] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[21] Paul Mc Namara, Rudy R Negenborn, Bart De Schutter, and Gordon Lightbody. Weight optimisation for iterative distributed model predictive control applied to power networks. *Engineering Applications of Artificial Intelligence*, 2012.

[22] P. Mhaskar, N.H. El-Farra, and P.D. Christofides. Predictive control of switched nonlinear systems with scheduled mode transitions. *IEEE Transactions on Automatic Control*, 50(11):1670–1680, 2005.

[23] H. Michalska and D.Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38(11):1623–1633, 1993.

[24] S.J. Qin and T.A. Badgwell. A survey of industrial model predictive control technology. *Control engineering practice*, 11(7):733–764, 2003.

[25] R. Scattolini. Architectures for distributed and hierarchical model predictive control-a review. *Journal of Process Control*, 19(5):723–731, 2009.

[26] Brett T Stewart, Stephen J Wright, and James B Rawlings. Cooperative distributed model predictive control for nonlinear systems. *Journal of Process Control*, 21(5):698–704, 2011.

[27] M. Vaccarini, S. Longhi, and M.R. Katebi. Unconstrained networked decentralized model predictive control. *Journal of Process Control*, 19(2):328–339, 2009.

[28] A. Venkat, J. Rawlings, and S. Wright. *Distributed model predictive control of large-scale systems*. Springer, 2007.

[29] Aleksandar Zecevic and Dragoslav D Siljak. *Control of complex systems: Structural constraints and uncertainty*. Springer, 2010.

[30] Y. Zheng, N. Li, and S. Li. Hot-rolled strip laminar cooling process plant-wide temperature monitoring and control. *Control Engineering Practice*, 21(1):23–30, 2013.

[31] Y Zheng and S. Li. The stabilization of coordinated distributed model predictive control. In *Large scale complex systems theory and applications*, volume 13, pages 182–187, 2013.

[32] Y. Zheng, S. Li, and N. Li. Distributed model predictive control over network information exchange for large-scale systems. *Control Engineering Practice*, 19:757–769, 2011.

[33] Y. Zheng, S. Li, and H. Qiu. Networked coordination-based distributed model predictive control for large-scale system. *IEEE Transactions on Control Systems Technology*, 21(3):991–998, 2013.

[34] Y. Zheng, S. Li, and X. Wang. Distributed model predictive control for plant-wide hot-rolled strip laminar cooling process. *Journal of Process Control*, 19(9):1427–1437, 2009.

[35] Y. Zheng, S. Li, and X. Wang. Horizon-varying model predictive control for accelerated and controlled cooling process. *Industrial Electronics, IEEE Transactions on*, 58(1):329–336, 2011.

[36] Y. Zheng, S. Li, and X. Wang. Optimization target resetting distributed model predictive control for accelerated cooling process. In *Intelligent Control and Automation (WCICA), 2012 10th World Congress on*, pages 2764–2769. IEEE, 2012.